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LETTER TO THE EDITOR

CLEBSCH–GORDAN COEFFICIENTS FOR THE QUANTUM ALGEBRA $\mathrm{SU}(2)_{p,q}$

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The Clebsch–Gordan coefficients for the $\mathrm{SU}(2)_{p,q}$ algebra are calculated using the covariant – tensor method for quantum groups. It is shown that the C.-G. coefficients depend on a single parameter $Q = \sqrt{pq}$.

During the past few years, much attention has been paid to the quantum deformations of Lie algebras (quantum groups)[1)], both from the mathematical and physical point of view. The main idea of physical application of the quantum groups is a generalization of the concept of symmetry. For example, the rules for the addition of angular momenta in q-deformed $\mathrm{SU}(2)_q$ algebra are generalized in accordance with q-deformed algebra and co-algebra[2)].

Multiparameter deformations of Lie algebras (with more than one deforming parameter) were also studied[3)].

In this Letter we calculate for the Clebsch–Gordan coefficients for two-parameter (p, q) deformed $\mathrm{SU}(2)_{p,q}$ algebra. We show that the C.-G. coefficients depend effectively on only one parameter $Q = \sqrt{pq}$. Our result are in agreement with Drinfeld and Reshetikhin's theorem[4)].

We recall the $\mathrm{SU}(2)_{p,q}$ algebra defined in references [3] and [5] (p and q are real parameters):

$$[J_0, J_\pm] = \pm J_\pm$$
$$[J_+, J_-]_{p,q} = J_+J_- - qp^{-1}J_-J_+ = [2J_0]_{p,q}$$
$$[2J_0]_{p,q} = \frac{p^{2J_0} - q^{-2J_0}}{p - q^{-1}}$$

$$\begin{aligned}(J_0)^\dagger &= J_0\\ (J_\pm)^\dagger &= J_\mp\ .\end{aligned} \tag{1}$$

The coproduct Δ is:

$$\begin{aligned}\Delta(J_\pm) &= J_\pm \otimes p^{J_0} + q^{-J_0} \otimes J_\pm\\ \Delta(J_0) &= J_0 \otimes 1 + 1 \otimes J_0\ .\end{aligned} \tag{2}$$

The finite dimensional unitary irreducible representation (IRREP) D^j of spin j contains the highest weight vector $|jj\rangle$, satisfying

$$\begin{aligned}J_0|jj\rangle &= j|jj\rangle\\ J_+|jj\rangle &= 0\\ \langle jj|jj\rangle &= 1\ .\end{aligned} \tag{3}$$

The other orthonormalized states of IRREP D^j, $|jm\rangle$, with $-j \le m \le j$, satisfy

$$\begin{aligned}J_+|jm\rangle &= \left(\frac{q}{p}\right)^{\frac{1}{2}(j-m-1)} \sqrt{[j-m]_{p,q}[j+m+1]_{p,q}}|j\,m+1\rangle\\ J_-|jm\rangle &= \left(\frac{q}{p}\right)^{\frac{1}{2}(j-m)} \sqrt{[j-m]_{p,q}[j-m+1]_{p,q}}|j\,m-1\rangle\\ J_0|jm\rangle &= m|jm\rangle\ .\end{aligned} \tag{4}$$

We calculate the C.-G. coefficients for the $\mathrm{SU}(2)_{p,q}$ quantum algebra using the covariant – tensor method recently proposed by us[6)]. The main results are written in tensor notation. The basis vectors in the tensor space $(V_2)^{\otimes k}$ are $|e_{a_1} \dots e_{a_k}\rangle$, with $a_1, \dots, a_k = 1, 2$.

Then

$$|jm\rangle = |e_{a1,\dots,a_k}\rangle = \frac{1}{\sqrt{f}} q^{-\frac{M}{2}} \sum_{\mathrm{perm}(a_1\dots a_k)} (pq)^{\frac{1}{2}\chi(a_1\dots a_k)} |ee_{a_1}\dots e_{a_k}\rangle \tag{5}$$

where the curly bracket $\{a_1 \dots a_k\}$ denotes the q-symmetrization. The summation runs over all the allowed permutations of the fixed set of indices (n_1 $1's$ and n_2 $2's$). $\chi(a_1 \dots a_k)$ is the number of inversions with respect to the normal order $11\dots122\dots2$, and

$$\begin{aligned} M &= n_1 n_2 = (j+m)(j-m) \\ j &= \frac{1}{2}(n_1+n_2) \quad m = \frac{1}{2}(n_1-n_2) \\ f &= \begin{pmatrix} 2j \\ j+m \end{pmatrix}_{p,q} = \frac{[2j]_{p,q}!}{[j+m]_{p,q}![j-m]_{p,q}!}\,. \end{aligned} \tag{6}$$

The important relations are:

$$\begin{aligned} f &= q^{-M} \sum_{\mathrm{perm}(a_1\dots a_k)} (pq)^{\chi(a_1\dots a_k)} \\ [n]_{p,q} &= \frac{p^n - q^{-n}}{p - q^{-1}} = \left(\frac{p}{q}\right)^{\frac{1}{2}(n-1)} [n]_Q \end{aligned} \tag{7}$$

with $Q = \sqrt{pq}$.

The dual states are

$$\begin{aligned} \langle ce_{a_1}\dots e_{a_k}| &= (|e_{a_1}\dots e_{a_k}\rangle)^\dagger \\ \langle jm| = (|jm\rangle)^\dagger &= \frac{1}{\sqrt{f}} q^{-\frac{M}{2}} \sum_{\mathrm{perm}(a_1\dots a_k)} Q^{\chi(a_1\dots a_k)} \langle e_{a_k}\dots e_{a_1}|. \end{aligned} \tag{8}$$

From the orthonormal condition

$$\langle e_{a_k}\dots e_{a_1}|e_{b_1}\dots e_{b_k}\rangle = \delta_{a_1b_1}\dots\delta_{a_kb_k} \tag{9}$$

and equation (7) it follows that $\langle jm_1|jm_2\rangle = \delta_{m_1m_2}$.

Applying $\Delta(J_\pm)$ and $\Delta(J_0)$ from equations (2), we obtain equations (4). It is important to note that $|jm\rangle_{p,q} = |jm\rangle_Q$.

Furthermore, the quadratic form, invariant under the action of the coproduct Δ (equations (2)), is

$$I = |e_{a_k}\dots e_{a_1}\rangle|e_{b_1}\dots e_{b_k}\rangle\varepsilon_{a_1b_1}\dots\varepsilon_{a_kb_k} \tag{10}$$

where

$$\varepsilon = \begin{pmatrix} 0 & p^{\frac{1}{2}} \\ -q^{-\frac{1}{2}} & 0 \end{pmatrix}$$

$$\left(\frac{\varepsilon}{\sqrt{[2]}}\right)_{p,q} = \left(\frac{\varepsilon}{\sqrt{[2]}}\right)_Q$$

$$\varepsilon_{ab}\varepsilon_{bc} = -\sqrt{\frac{p}{q}}\delta_{ac} \qquad \varepsilon_{ab}\varepsilon_{cb} = \sqrt{\frac{p}{q}}(Q^{2J_0})_{ac}$$

$$\varepsilon_{ab}\varepsilon_{ab} = [2]_{p,q} \qquad (\varepsilon_{ba})_{p,q} = -(\varepsilon_{ab})_{q^{-1},p^{-1}}. \tag{11}$$

The general form of the C.-G. coefficients for the $SU(2)_{p,q}$ algebra is[6)]

$$\begin{aligned} &\langle j_1 m_1 j_2 m_2 | JM \rangle_{p,q} = N_{p,q} \cdot F(p,q) = \\ &= N_{p,q} \sum_{m=-j}^{+j} \langle j_1 m_1 | (j_1 - j)(m_1 - m) j m \rangle_{p,q} \times \\ &\times \langle j_2 m_2 | j - m (j_2 - j)(m_2 + m) \rangle_{p,q} \langle j \, m \, j - m | 00 \rangle_{p,q} \times \\ &\times \langle (j_1 - j)(m_1 - m)(j_2 - j)(m_2 + m) | JM \rangle_{p,q} \end{aligned} \tag{12}$$

where $2j = j_1 + j_2 - J$ and $N_{p,q}$ is the norm depending on j_1, j_2 and J.

The C.-G. coefficients are real for p, q real and the following relation is valid:

$$\langle j_1 m_1 j_2 m_2 | JM \rangle_{p,q} = \langle JM | j_1 m_1 j_2 m_2 \rangle_{p,q}. \tag{13}$$

Using the tensor notation $|jm\rangle = |e_{a_1 \ldots a_k)}\rangle$, $k = 2j$, we first calculate C.-G. coefficients for $j_1 \otimes j_2 \to j_1 + j_2$:

$$\begin{aligned} &\langle j_1 + j_2 \, m_1 + m_2 | j_1 m_1 j_2 m_2 \rangle_{p,q} = \langle e_{\{b,a\}} | e_{\{a\}} e_{\{b\}} \rangle_{p,q} = \\ &= \sqrt{\left(\frac{f_1 \cdot f_2}{f_3}\right)_{p,q}} \cdot \left(\frac{q}{p}\right)^{\frac{1}{4}(M_1 + M_2 - M_3)} (p \cdot q)^{\frac{1}{2}(j_1 m_2 - j_2 m_1)} = \\ &= \sqrt{\left(\frac{f_1 \cdot f_2}{f_3}\right)_Q} Q^{j_1 m_2 - j_2 m_1} = \langle j_3 m_3 | j_1 m_1 j_2 m_2 \rangle_Q \end{aligned} \tag{14}$$

where

$$M_i = (j_i + m_i)(j_i - m_i), \quad f_1 = \begin{pmatrix} 2j_i \\ j_i + m_i \end{pmatrix}_{p,q} \quad j_3 = j_1 + j_2$$

,

$$m_3 = m_1 + m_2 \quad \text{for} \quad i = 1, 2, 3$$

.

We point out that these C.-G. coefficients depend effectively only on one parameter $Q = \sqrt{pq}$ and that

$$\langle j_1(\pm j_1)j_2(\pm j_2)|(j_1 + j_2) \pm (j_1 + j_2)\rangle_{p,q} = 1. \tag{15}$$

There of the four C.-G. coefficients appearing on the right-hand side of equation (12) have the simple form (14). The fourth coefficient $\langle jmj - m|00\rangle_{p,q}$ also has a simple form and depends only on one parameter Q. Namely, for $n = 2j$ we have

$$\begin{aligned}\langle jmj - m|00\rangle_{p,q} &= \frac{1}{\sqrt{[n+1]_{p,q}}}\, \varepsilon_{a_1 b_1} \ldots \varepsilon_{a_n b_n} = \\ &= (-1)^{j-m} \frac{1}{\sqrt{[n+1]_Q}}\, Q^m = \langle jmj - m|00\rangle_Q.\end{aligned} \tag{16}$$

After inserting equations (16) and (14) into equation (12), we conclude that the C.-G. coefficients depend only on one parameter Q:

$$\begin{aligned}&\langle j_1 m_1 j_2 m_2|JM\rangle_{p,q} = \langle j_1 m_1 j_2 m_2|JM\rangle_Q = \\ &= N_Q \sum_{m=-j}^{+j} \frac{(-1)^{j-m}}{\sqrt{[2j+1]_Q}} \cdot Q^{(j_1 m_2 - j_2 m_1)} Q^{m(2J+2j+1)} \times \\ &\times \frac{\begin{pmatrix} 2j \\ j+m \end{pmatrix}_Q \cdot \begin{pmatrix} 2j_1 - 2j \\ j_1 - j + m_1 - m \end{pmatrix}_Q \cdot \begin{pmatrix} 2j_2 - 2j \\ j_2 - j + m_2 - m \end{pmatrix}_Q}{\sqrt{\begin{pmatrix} 2J \\ J+M \end{pmatrix}_Q \cdot \begin{pmatrix} 2j_1 \\ j_1 + m_1 \end{pmatrix}_Q \cdot \begin{pmatrix} 2j_2 \\ j_2 + m_2 \end{pmatrix}_Q}}\end{aligned} \tag{17}$$

with $2j = j_1 + j_2 - J$ and the norm

$$\begin{aligned}N_{p,q} &= \sqrt{\frac{[2j_1]_{p,q}![2j_2]_{p,q}![2J+1]_{p,q}![j_1+j_2-J+1]_{p,q}!}{[j_1+j_2-J]_{p,q}![j_1-j_2+J]_{p,q}![-j_1+j_2+J]_{p,q}![j_1+j_2+J+1]_{p,q}!}} \\ &\equiv N_Q.\end{aligned} \tag{18}$$

Finally, we mention that the C.-G. problem for the two-parameter quantum algebra $SU(2)_{p,q}$ was also analyzed in refernce [5] using the projection operator technique. However, their calculation contains a few errors, for example the expression for their projection operator $p^j_{mm'} = |jm\rangle\langle jm'|$ is wrong and their C.-G. coefficients do not satisfy orthonormality relations. Hence, the conclusion that C.-G. coefficients nontrivialy depend on both parameters p and q is not correct[7].

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CLEBSCH–GORDANOVI KEOFICIJENTI ZA KVANTNU ALGEBRU SU(2)$_{p,q}$

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Clebsch–Gordanovi koeficijenti SU(2)$_{p,q}$ algebre izračunati su pomoću kovarijantne tenzorske metode za kvantne grupe. Pokazano je da Clebsch–Gordanovi koeficijenti ovise o jednom parametru $Q = \sqrt{pq}$.